# Some Applications of Partial Orderings to Iterative Methods for Rectangular Linear Systems

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ABSTRACT

In this paper we extend the notions of $K$-semipositivity, $K$-monotonicity and of $K$-positive subinverses, introduced by Vandergraft for square matrices, to rectangular matrices. We apply these extensions to iterative methods for linear and rectangular systems $Ax=b$, $A\in R^{m\times n}$, induced by subproper and proper splittings of the matrix $A$.

## 1. INTRODUCTION

Let $A$ be a real $m\times n$ matrix, and consider the system of linear equations

$$Ax=b, \tag{1.1}$$

where $x$ is a real $n$-vector and $b$ is a real $m$-vector.

In recent years several authors have considered the problem of obtaining iterative methods which converge to a solution of (1.1) when the system is solvable, or to an approximate solution to (1.1) when the system is not solvable, (e.g., [1, 2, 3, 10, 12, 13]). The approaches used there employ the generalized inverse of a matrix. Consequently, these papers include schemes which are designed to converge to the minimum $l_2$-norm solution to (1.1) when the system is solvable, and to a least-squares solution and/or to the minimum $l_2$-norm least-squares solution to (1.1) when the system is not solvable.

For an $m \times n$ matrix $A$ the splitting into

$$A = M - Q \tag{1.2}$$

is *subproper* [13] if

$$R(A) \subseteq R(M) \quad \text{and} \quad N(M) \subseteq N(A), \tag{1.3}$$

where $R(A)$ and $N(A)$ denote the range and null space of $A$, respectively. The splitting (1.2) of $A$ is *proper* [3] if equalities hold in (1.3). The splitting (1.2) leads to the iteration

$$x_i = M^+Qx_{i-1} + M^+b, \tag{1.4}$$

where $M^+$ denotes the Moore-Penrose generalized inverse of $M$. Theorem 4 in [1] and Theorem 1 in [13] can be amalgamated to show the following convergence result.

THEOREM 0. *Let* (1.2) *be a subproper splitting for $A$ in* (1.1). *Then the subspaces $R(A)$ and $N(AM^+)$ are complementary in $R^m$ and the sequence*

$$\{x_i - iM^+b_{N(AM^+),R(A)}\} \tag{1.5}$$

*converges to a solution of the system*

$$Ax = b_{R(A),N(AM^+)} \tag{1.6}$$

*for all $x_0$ if and only if the Jordan blocks of $M^+Q$ corresponding to the eigenvalue $\lambda = 1$ are all of order* 1, *and all the eigenvalues of $M^+Q$ different from* 1 *are in modulus smaller than* 1. *Here $x_i$ is given by* (1.4), *and $b_{N(AM^+),R(A)}$ and $b_{R(A),N(AM^+)}$ denote the projections of $b$ on $N(AM^+)$ along $R(A)$ and on $R(A)$ along $N(AM^+)$, respectively.*

The (*iteration*) matrix $M^+Q$ possessing the properties of Theorem 0 is called $s$-convergent for $A$ [13]. Some important facts concerning Theorem 0 are as follows: (i) If $b \in R(A)$ or the splitting (1.2) is proper, then the sequence (1.5) reduces to the sequence of iterates generated by (1.4). (ii) The results of Theorem 0 contain many well-known special cases. For a detailed account of these cases see [1] and [13]. (iii) A characterization for the splitting (1.2) to be proper is given in [2]. Several characterizations for the splitting (1.2) to be subproper are given in [9].

The object of this paper is to extend the applicability of the concept of *K-semipositivity* of a real square matrix (see below), due to Vandergraft [18], to the study of iterative schemes induced by subproper and proper splittings of $A$ in (1.1).

To achieve this goal we require the following definitions.

DEFINITION 1. Let $B\in R^{p\times q}$, the space of real $p\times q$ matrices, and let $K$ and $L$ be solid cones in $R^q$ and $R^p$, where $R^s\equiv R^{s\times 1}$, respectively.

(i) A matrix $C\in R^{q\times p}$ is an *$L$-right generalized subinverse* ($L$-r.g.s.i.) of $B$ if

$$(BB^+ - BC)(L)\subseteq L.$$

(ii) A matrix $C\in R^{q\times p}$ is a *$K$-left generalized subinverse* ($K$-l.g.s.i.) of $B$ if

$$(B^+B - CB)(K)\subseteq K. \tag{1.7}$$

DEFINITION 2. Let $B\in R^{p\times q}$, and let $K\subseteq R^q$ and $L\subseteq R^p$ be solid cones. $B$ is said to be *$(K,L)$-semipositive* if

$$B(K^0)\cap L^0\neq\varnothing, \tag{1.8}$$

where $K^0$ and $L^0$ denote the interiors of $K$ and $L$, respectively. $B$ is said to be *$(K,L)$-weakly semipositive* if

$$B(K^0)\cap L\neq\varnothing.$$

When $q=p$, (1.7) does not necessarily reduce to Vandergraft's concept [18] "$C$ is a $K$-positive left subinverse of $B$", as will be shown in Sec. 1.1 on notation and preliminaries. The condition (1.8) is an extension to Vandergraft's concept of $K$-semipositivity. For, in the language of Vandergraft, if $q=p$, $K=L$ and (1.8) holds, then $B$ is *$K$-semipositive.*

DEFINITION 3. Let $B\in R^{p\times q}$, and let $K\subseteq R^q$ and $L\subseteq R^p$ be solid cones. Then $B$ is said to be *$(K,L)$-monotone* if

$$B^+(L)\subseteq K.$$

In Sec. 2 we study the relationships between these concepts and characterize some of them. The results in this section extend to the general $p\times q$ case the results of Vandergraft [18] for the case where $B$ is square and nonsingular. In Sec. 3 we obtain conditions for $\rho(BB^+ - BC)$, the spectral radius of $BB^+ - BC$, to be less than or equal to 1 when $C$ is an $L$-r.g.s.i. of $B$. We show that when $\lambda = 1$ is an eigenvalue of $BB^+ - BC$, these conditions imply that the Jordan blocks corresponding to $\lambda = 1$ are all of order 1. The section is concluded with a result stating sufficient conditions for the powers of the matrix $BB^+ - BC$ to converge to some matrix. The results of Sec. 3 are applied in Sec. 4 to iterative schemes for (1.1) induced by subproper splittings and proper splittings of the matrix $A$.

### 1.1. *Notation and Preliminaries*

For $B \in R^{p\times q}$, $B^T$ denotes the transpose of $B$. $P_{R(B)}$ denotes the orthogonal projector on $R(B)$.

For $B \in R^{p\times q}$, $B^+$ is the unique matrix $X \in R^{q\times p}$ satisfying the matrix equations $B = BXB$, $X = XBX$, $BX = (BX)^T$ and $XB = (XB)^T$. Thus $BB^+ = P_{R(B)}$, $B^+B = P_{R(B^T)}$, $R(B^+) = R(B^T)$ and $N(B^+) = N(B^T)$.

A closed subset $K \subseteq R^q$ is a *cone* if $\alpha K \subseteq K$, $\alpha \geqslant 0$, $K + K \subseteq K$ and $K \cap (-K) = \{0\}$. A cone $K$ is *solid* if $K^0 \neq \varnothing$; it is *reproducing* if $K - K = R^q$. In $R^q$ a cone $K$ is reproducing if and only if it is solid. A solid cone $K$ induces a partial ordering of $R^q$ given by $x \overset{K}{\leqslant} y$ if and only if $y - x \in K$, and $x \overset{K}{<} y$ if and only if $y - x \in K^0$. An important fact is that if $k_0 \in K^0$ and $y \in R^q$, then for some $\alpha > 0$, $k_0 - \alpha y \in K$. For $B \in R^{q\times q}$ we shall use the notation $B \overset{K}{\geqslant} 0$ if $Bk \in K$ whenever $k \in K$. For $x \in R^q$ and $B \in R^{q\times q}$ we shall use the notation $x \geqslant 0$ and $B \geqslant 0$ to denote a vector and a matrix with nonnegative entries, respectively. By $R^q_+$ we denote the nonnegative orthant in $R^q$. $R^q_+$ is a solid cone in $R^q$.

For $B \in R^{q\times q}$, Vandergraft [18] defines a matrix $C \in R^{q\times q}$ to be a *$K$-positive left subinverse* of $B$ if $I \overset{K}{\geqslant} CB$ for some solid cone $K \subseteq R^p$. Then 0 is a $K$-positive left subinverse for any $B \in R^{q\times q}$ and for any solid cone $K \subseteq R^q$. But 0 is not always a $K$-l.g.s.i. for $B \in R^{q\times q}$, as the following example shows. Let $L = R^2_+$, and let

$$B = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

Then

$$P_{R(B^T)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \not\geqslant 0.$$

Other examples of matrices $B, C \in R^{q\times q}$ such that $C$ is a $K$-l.g.s.i. of $B$ and such that $C$ is not a $K$-positive left subinverse of $B$, or such that $C$ is a $K$-positive left subinverse of $B$ but $C$ is not a $K$-l.g.s.i. of $B$, are easy to construct and are omitted here. However, we note that if $p \geqslant q$ and $B \in R^{p\times q}$ is of full column rank, then $C$ is a $K$-l.g.s.i. of $B$ if and only if $I \overset{K}{\geqslant} CB$.

Finally, in Sec. 3 use will be made of the following result of Rheinboldt and Vandergraft [16, Theorems 3, 4].

LEMMA 0. *Let $B \in R^{p\times p}$, and let $L \subseteq R^p$ be a solid cone such that $B \overset{L}{\geqslant} 0$. Then*:

(i) *If $Bl_0 \overset{L}{\leqslant} \alpha l_0$ for some $l_0 \in L^0$, then $\rho(B) \leqslant \alpha$.*

(ii) *$Bl_0 \overset{L}{<} \alpha l_0$ for some $l_0 \in L^0$ if and only if $\rho(B) < \alpha$.*

## 2. LINEAR TRANSFORMATIONS BETWEEN CONES AND BETWEEN PARTS OF CONES

Matrices (or linear transformations) which leave cones in $R^q$ invariant have been studied by various authors, e.g., [6, 17, 19]. Here a key result is that if $B \overset{K}{\geqslant} 0$, where $B \in R^{q\times q}$ and $K \subseteq R^q$ is a solid cone, then $K$ contains an eigenvector corresponding to $\rho(B)$. In connection with convergence theory for iterative methods for linear (and nonsingular) systems, of great importance are conditions which ensure that $B$ is *$K$-monotone*, i.e., $B^{-1} \overset{K}{\geqslant} 0$. With respect to the latter point, Vandergraft [18] introduced the notion of $K$-semipositivity and established its connections to $K$-monotonicity and also to positive definiteness. We now extend some of his results to the (general) rectangular case.

LEMMA 1. *Let $B \in R^{p\times q}$ and let $K \subseteq R^q$ and $L \subseteq R^p$ be solid cones. Then the following statements are equivalent*:

(i)

$$B(K^0) \subseteq L^0. \tag{2.1}$$

(ii) *$B(K) \subseteq L$, and there exists a vector $x \in R^q$ such that*

$$Bx \in L^0. \tag{2.2}$$

(iii) *$B(K) \subseteq L$, and $P_{R(B),S}$ is $L$-semipositive, where $P_{R(B),S}$ is a projection on $R(B)$ along an arbitrary complementary subspace $S$ [of $R(B)$] in $R^p$.*

*Proof.* That (i) implies (ii) is obvious. (ii) implies (i): Let $k_0 \in K^0$, and let $\alpha > 0$ be a number such that $k_0 - \alpha x \in K$. This implies

$$Bk_0 - \alpha Bx = B(k_0 - \alpha x) \in L,$$

so that $Bk_0 \overset{L}{\geqslant} \alpha Bx \overset{L}{>} 0$ and hence $Bk_0 \in L^0$. (ii) implies (iii) because $P_{R(B),S}Bx = Bx \in L^0$. (iii) implies (ii): Let $l_0 \in L^0$ be a vector such that $P_{R(B),S}l_0 \in L^0$. Then for $x \equiv B^+ P_{R(B),S} l_0 \in R^q$, $Bx = P_{R(B),S} l_0 \in L^0$. ■

COROLLARY 1 (Vandergraft [18]). *Let $K \subseteq R^q$ be a solid cone, and suppose that $B \in R^{q \times q}$ is nonsingular. Then $B \overset{K}{\geqslant} 0$ if and only if $B(K^0) \subseteq K^0$.*

*Proof.* For some $k_0 \in K^0$, set $x \equiv B^{-1}k_0$. Then $Bx \in K^0$ and the condition (2.2) is always satisfied. ■

A weakening of the condition (2.1) is that $B$ is $(K,L)$-semipositive. Some conditions which ensure $(K,L)$-semipositivity are given next.

LEMMA 2. *Let $B \in R^{p \times q}$, and let $K \subseteq R^q$ and $L \subseteq R^p$ be solid cones. Then:*

(i) *$B$ is $(K,L)$-semipositive if and only if*

$$Bk \in L^0 \tag{2.3}$$

*for some $k \in K$.*

(ii) *$B$ is $(K,L)$-semipositive if $B$ is $(K,L)$-monotone and there exists a vector $x \in R^q$ such that (2.2) holds.*

(iii)

$$B\left[K^0 \cap R(B^T)\right] \cap L^0 \neq \varnothing, \tag{2.4}$$

*if and only if*

$$B^+\left[L^0 \cap R(B)\right] \cap K^0 \neq \varnothing.$$

*Proof.* The proof of part (i) is obvious and is thus omitted.

(ii) Let $k_0 \in K^0$. Because $Bx \in L^0$, we have $Bx - \alpha Bk_0 \in L$ for some $\alpha > 0$. Hence, $B^+Bx - \alpha B^+Bk_0 \in K$, since $B$ is $(K,L)$-monotone. Set $x_1 \equiv \alpha k_0 + (B^+Bx - \alpha B^+Bk_0)$. Then $x_1 \in K^0$ and $Bx_1 = Bx \in L^0$.

(iii) Assume (2.4). Then there exists a vector $k_0 \in K^0 \cap R(B^T)$ such that $Bk_0 \in L^0 \cap R(B)$ and also $B^+(Bk_0) = k_0$. The converse follows in a similar manner. ∎

REMARKS. (1) In Sec. 3 we show that in some special cases the condition (2.3) can be weakened further to yield necessary and sufficient conditions for $B$ to be $(K,L)$-semipositive.

(2) If $B$ is $(K,L)$-monotone and there exists a vector $x \in R^q$ such that $Bx \in L$, then $B$ need not be $(K,L)$-weakly semipositive.

(3) If there exists a vector $x \in R^q$ such that (2.2) holds but $B$ is not $(K,L)$-monotone, then $B$ need not be $(K,L)$-semipositive.

COROLLARY 2 (Vandergraft [18]). *Let $K \subseteq R^q$ be a solid cone, and suppose $B \in R^{q \times q}$ is nonsingular. Then*

(i) *$B$ is $K$-semipositive if $B$ is $K$-monotone.*
(ii) *$B$ is $K$-semipositive if and only if $B^{-1}$ is $K$-semipositive.*

Necessary and sufficient conditions for $B \in R^{p \times q}$ to be $(K,L)$-monotone are contained in the next lemma.

LEMMA 3. *Let $B \in R^{p \times q}$, and let $K \subseteq R^q$ and $L \subseteq R^p$ be solid cones. Then the following statements are equivalent:*

(i) *$B$ is $(K,L)$-monotone.*
(ii) *There exists a matrix $C_1 \in R^{q \times p}$ such that*

$$R(C_1) \subseteq R(B^T), \tag{2.5}$$

$$P_{R(B)} \overset{L}{\geqslant} BC_1 \tag{2.6}$$

*(i.e., $C_1$ is an L-r.g.s.i. of B),*

$$\rho(P_{R(B)} - BC_1) < 1 \tag{2.7}$$

*and*

$$C_1(L) \subseteq K. \tag{2.8}$$

(iii) *There exists a matrix* $C_2 \in R^{q\times p}$ *such that*

$$N(B^T) \subseteq N(C_2), \tag{2.9}$$

$$P_{R(B^T)} \overset{K}{\geqslant} C_2 B \tag{2.10}$$

*(i.e., $C_2$ is a K-l.g.s.i. of B),*

$$\rho(P_{R(B^T)} - C_2 B) < 1, \tag{2.11}$$

$$C_2(L) \subseteq K. \tag{2.12}$$

(iv) $Bx \in P_{R(B)}L,\ x \in R(B^T) \Rightarrow x \in K.$

*Proof.* The proof that (i) and (iv) are equivalent is analogous to the proof of [4, Theorem 2], where the special case $K = R^q_+$ and $L = R^p_+$ is considered.[1] (i) implies (ii) and (iii) by choosing $C_1 = C_2 = B^+$. (ii) implies (i): The conditions (2.6) and (2.7) show that

$$0 \overset{L}{\leqslant} \sum_{i=0}^{\infty} (P_{R(B)} - BC_1)^i = (I - P_{R(B)} + BC_1)^{-1}. \tag{2.13}$$

Now $B^+BC_1 = C_1$ by (2.5), and so

$$B^+(I - P_{R(B)} + BC_1) = C_1. \tag{2.14}$$

Thus since $I - P_{R(B)} + BC_1$ is nonsingular, one obtains from (2.14) the relationship

$$B^+ = C_1(I - P_{R(B)} + BC_1)^{-1}.$$

But then (2.13) and (2.8) show that $B$ is $(K,L)$-monotone. The proof that (iii) implies (i) is similar to the proof that (ii) implies (i). ∎

Notice that (2.14) implies

$$R(B^T) = R(C_1). \tag{2.15}$$

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[1]More recently, Carlson [8] has shown the equivalence of (i) and (iv) for general subsets $K \subseteq R^q$ and $L \subseteq R^p$, respectively.

Likewise, the conditions of part (iii) of the above theorem imply

$$N(B^T)=N(C_2). \tag{2.16}$$

COROLLARY 3 (Price [15]). *Suppose $B\in R^{q\times q}$ is nonsingular. Then $B$ is monotone (i.e., $B^{-1}\geqslant 0$) if and only if there exists a nonsingular matrix $C\in R^{q\times q}$ such that $C\geqslant 0$, $I\geqslant CB$ and $\rho(I-CB)<1$.*

Lemma 3 can also be used to establish the necessity part of the next corollary.

COROLLARY 4 (Berman and Plemmons [3]). *Let $K\subseteq R^n$ and $L\subseteq R^m$ be solid cones, and let (1.2) be a proper splitting for $A\in R^{m\times n}$ such that*[2]

$$M^+(L)\subseteq K \tag{2.17}$$

*and such that*

$$M^+Q(K)\subseteq K. \tag{2.18}$$

*Then a necessary and sufficient condition for*

$$\rho(M^+Q)<1 \tag{2.19}$$

*is that*

$$A^+(L)\subseteq K. \tag{2.20}$$

*Proof* (for the necessity part only). Since (1.2) is a proper splitting for $A$, we have

$$N(A^T)=N(M^+). \tag{2.21}$$

Moreover, by (2.18) and (1.2) we have

$$A^+A-M^+A\overset{K}{\geqslant}0.$$

[2]A splitting (1.2) for $A$ satisfying conditions (2.17) and (2.18) is called a $(K,L)$-*weakly regular splitting* [2].

Substituting $q=n$, $p=m$, $B=A$ and $C_2=M^+$ in Lemma 3(iii), we observe that the conditions (2.21), (2.17), (2.18) and (2.19) satisfy, respectively, the conditions (2.9), (2.12), (2.10) and (2.11), and hence $A^+(L)\subseteq K$. ■

On the other hand, Corollary 4 yields the following result.

COROLLARY 5. *Let $K\subseteq R^q$ and $L\subseteq R^p$ be solid cones, and suppose $C_2\in R^{q\times p}$ is a K-l.g.s.i. for $B\in R^{p\times q}$ such that* (2.15), (2.16) *and* (2.12) *hold. If $B$ is $(K,L)$-monotone, then*

$$\rho(P_{R(B^T)}-C_2B)<1.$$

*Proof.* The proof follows at once by Corollary 4, since the splitting

$$B=C_2^+-(C_2^+-B)$$

is proper. ■

Note that under the conditions of Corollary 5 on $B$ and $C_2$, $B$ is a $K$-r.g.s.i. of $C_2$. In the next section, with slight variation in notation, one of our results shows that some of the conditions of Corollary 5 can be replaced by other conditions which assure (2.11).

## 3. THE MAIN RESULTS

In this section, whenever convenient, we shall denote by $E$ the matrix

$$E=P_{R(B)}-BC, \tag{3.1}$$

where $B\in R^{p\times q}$ and $C\in R^{q\times p}$, and we shall use $\rho$ to denote $\rho(E)$.

THEOREM 1. *Let $B\in R^{p\times q}$, let $L\subseteq R^q$ be a solid cone, and let $C\in R^{q\times p}$ be an L-r.g.s.i. of B. Suppose there exists a vector $l\in L$ such that*

$$z\equiv P_{R(B)}l\in L^0 \tag{3.2}$$

*and such that*

$$BCz\in L. \tag{3.3}$$

*Then*:

(i) $\rho \leqslant 1$.

(ii) *If* $\rho = 1$ *so that* $\lambda = 1$ *is an eigenvalue of* $E$, *then the Jordan blocks of* $E$ *corresponding to* $\lambda = 1$ *are all of order* 1.

*Proof.*

(i) The assumption that $C$ is an $L$-r.g.s.i. of $B$, (3.3), (3.2) and (3.1) show that

$$Ez \overset{L}{\leqslant} z. \tag{3.4}$$

Since $z \in L^0$, $\rho \leqslant 1$ by Lemma 0.

(ii) We first show that for each $x \in R^p$, the sequence $\{E^i x\}$ is bounded. Since $L$ is a reproducing cone, it suffices to show that for each $l_1 \in L$, the sequence $\{E^i l_1\}$ is bounded. Now, induction on (3.4) gives

$$E^i z \overset{L}{\geqslant} E^{i+1} z \overset{L}{\geqslant} 0, \qquad i = 0, 1, \ldots .$$

Thus $\{E^i z\}$ is a bounded (and in fact convergent) sequence. For $l_1 \in L$, let $\alpha > 0$ be a number such that $z - \alpha l_1 \in L$, and resolve

$$z = \alpha l_1 + (z - \alpha l_1). \tag{3.5}$$

Since $E^i \overset{L}{\geqslant} 0$ for $i \geqslant 0$, we have from (3.5)

$$E^i z \overset{L}{\geqslant} \alpha E^i l_1 \overset{L}{\geqslant} 0, \qquad i = 0, 1, \ldots .$$

Thus $\{E^i l_1\}$ is a bounded sequence for $l_1 \in L$, so that $\{E^i x\}$ is a bounded sequence for $x \in R^p$.

Assume that some elementary divisor of $E$ corresponding to $\lambda = 1$ is not simple. Then there exists a principal eigenvector $w$ of $E$, $w \neq 0$, such that

$$Ew = w + v, \tag{3.6}$$

where $v \neq 0$ and $Ev = v$. Successive application of $E$ to (3.6) yields

$$E^i w = w + iv, \qquad i = 2, 3, \ldots .$$

But then $\{E^i w\}$ is not a bounded sequence, which is a contradiction. Thus the Jordan blocks of $E$ corresponding to $\lambda = 1$ are all of order 1. ■

REMARK. If $C$ is an $L$-r.g.s.i. of $B$ such that (3.2) holds and such that $\rho \leqslant 1$, then condition (3.3) need not be satisfied as we show now. Let

$$L = R_+^2, \qquad B = I \quad \text{and} \quad C = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus $E \overset{L}{\geqslant} 0$, $\rho \leqslant 1$, and (3.2) holds. But $BC(L^0) \cap L = \varnothing$.

One implication of Theorem 1 is the following result.

COROLLARY 6. *Under the conditions of Theorem 1, if*

$$R(C) \subseteq R(B^T) \tag{3.7}$$

*and*

$$R(B) \cap N(C) = \{0\}, \tag{3.8}$$

*then* $\rho < 1$.

*Proof.* By Theorem 1, $\rho \leqslant 1$. Assume that $\rho = 1$. Then for some $x \neq 0$,

$$(P_{R(B)} - BC)x = x. \tag{3.9}$$

Premultiplying both sides of (3.9) by $B^+$ and taking account of (3.7), we have

$$Cx = 0.$$

Because of (3.9), $x \in R(B)$, and so by (3.8) $x = 0$, which contradicts $x \neq 0$. Hence $\rho < 1$. ∎

Before considering other implications of Theorem 1, let us state our second main result, followed by a corollary due to Vandergraft.

THEOREM 2. *Let* $B \in R^{p \times q}$, *let* $L \subseteq R^p$ *be a solid cone, and let* $C \in$

*$R^{q\times p}$ be an L-r.g.s.i. of B such that*

$$N(B^T)\subseteq N(C). \tag{3.10}$$

*Then $\rho<1$ and $P_{R(B)}$ is L-semipositive if and only if BC is L-semipositive.*

*Proof.*

(Only if.) Since $\rho<1$ and $C$ is an $L$-r.g.s.i. of $B$, we have

$$0 \overset{L}{\leqslant} \sum_{i=0}^{\infty} E^i=(I-P_{R(B)}+BC)^{-1}. \tag{3.11}$$

Let $l_0\in L^0$ be a vector with

$$z_1\equiv P_{R(B)}l_0\in L^0.$$

Then (3.11) and Corollary 1 imply

$$z_2\equiv(I-P_{R(B)}+BC)^{-1}z_1\in L^0.$$

Now $(I-P_{R(B)}+BC)z_2=z_1$, and so premultiplying this equality on both sides by $P_{R(B)}$ yields

$$BCz_2=z_1,$$

so that $BC$ is $L$-semipositive.

*If.* Let $l_0\in L^0$ be a vector such that

$$w\equiv BCl_0\in L^0. \tag{3.12}$$

Since $E \overset{L}{\geqslant} 0$, (3.1) and (3.12) show that

$$w_1\equiv P_{R(B)}l_0=El_0+w\in L^0,$$

so that $P_{R(B)}$ is $L$-semipositive. Furthermore $C=CP_{R(B)}$ by (3.10); thus $w=BCw_1$, and so

$$Ew_1=w_1-w \overset{L}{<} w_1.$$

Hence $\rho<1$ by Lemma 0. ∎

COROLLARY 7 (Vandergraft [18]). *Let $L \subseteq R^p$ be a solid cone, and let $C \in R^{p\times p}$ satisfy $I \overset{L}{\geqslant} C$. Then $C$ is $L$-semipositive if and only if $\rho(I-C)<1$.*

*Proof.* For $B \equiv I = P_{R(B)}$, $C$ is an $L$-r.g.s.i. of $B$ such that (3.10) holds. ∎

We should remark here that in many applications of Corollary 7, $C$ is nonsingular. When this occurs we have that for $B=I$, the conditions (3.7) and (3.8) are satisfied. Thus, if $C\in R^{p\times p}$ is nonsingular and $I \overset{L}{\geqslant} C$, then Corollaries 6 and 7 together show that $\rho(I-C)<1$ if and only if $C$ is $L$-weakly semipositive. This slightly strengthens the conclusions of Corollary 7. A further amplification of this remark follows our next result.

THEOREM 3. *Let $B\in R^{p\times q}$, let $L\subseteq R^p$ be a solid cone, and let $C\in R^{q\times p}$ be an $L$-r.g.s.i. of $B$ such that*

$$R(C)=R(B^T) \tag{3.13}$$

*and such that*

$$N(C)=N(B^T). \tag{3.14}$$

*Assume further that*

$$P_{R(B)}(L^0)\subseteq L^0. \tag{3.15}$$

*Then the following statements are equivalent:*

(i) *$BC$ is $L$-semipositive.*
(ii) *$BC$ is $L$-weakly semipositive.*
(iii) *$BCl \in L^0$ for some $l\in L$.*

*Proof.* The conditions (3.13) and (3.14) imply (3.7), (3.8) and (3.10). We show (i)⇒(ii)⇒(i)⇒(iii)⇒(i). (i)⇒(ii) trivially. (ii) implies (i): Let $l_0\in L^0$ be a vector with $BCl_0\in L$. By (3.14) $BC=BCP_{R(B)}$ and by (3.16) $P_{R(B)}l_0\in L^0$. Thus all the assumptions of Corollary 6 are satisfied, and so $\rho<1$. The conclusion now follows by Theorem 2. (i) implies (iii) trivially. (iii) implies (i) by Lemma 2(i). ∎

REMARK. In [7] Bohl defines the notion of *Zeilensummenbedingung* (ZSB). Let $C=[c_{ij}]\in R^{p\times p}$ with $c_{ij}\leqslant 0$, $i\neq j$, $i,j=1,\dots,p$. Then $C$ is ZSB for $x\geqslant 0$ if:

(i) $Cx \geqslant 0$ and $\{i \in \tau | (Cx)_i > 0\} \neq \varnothing$, where $\tau \equiv \{1,\dots,p\}$ and $(Cx)_j$, $j \in \tau$, denotes the $j$th entry of $Cx$.

(ii) For each $i_0 \in \tau$ with $(Cx)_{i_0} = 0$ there exist indices $i_1,\dots,i_r \in \tau$ with $c_{i_k i_{k+1}} \neq 0$, $0 \leqslant k \leqslant r-1$, such that $(Cx)_{i_r} > 0$.

Bohl shows that if $C$ is ZSB for $x \geqslant 0$, then $x > 0$ and $c_{ii} > 0$, $i \in \tau$. Then, in the language of Varga [20], $C$ is of *generalized positive type* and also $C = \mathfrak{M}(C)$, *the comparison matrix* for $C$. The condition that $C$ is $R^p_+$-weakly semipositive is a weakening of the condition that $C$ is ZSB for some $x \geqslant 0$. Thus if we explicitly assume that $C$ is nonsingular, then Theorem 3 appears to strengthen the following result of Bohl [7, Satz 2.2], which we state here for the sake of completeness: If $C \in R^{p \times p}$ with $I - C \geqslant 0$, then the following statements are equivalent:

(i) $C$ is ZSB for some $x \geqslant 0$.
(ii) $Cx > 0$ for some $x > 0$.
(iii) $C^{-1} \geqslant 0$.

Under different conditions to those considered in this section we have the following theorem:

THEOREM 4. *Let $B \in R^{p \times q}$, let $L \subseteq R^p$ be a solid cone, and let $C \in R^{p \times q}$ be an L-r.g.s.i. of $B$ such that $E \overset{L}{\geqslant} 0$ and such that*

$$BCE \overset{L}{\geqslant} 0. \tag{3.16}$$

*Then the powers of $E$ converge to a projection on $N(I-E)$ along $R(I-E)$.*

*Proof.* By (3.1) $BC = P_{R(B)} - E$, and since $R(E) \subseteq R(B)$, we see by (3.16) that

$$BCE = E - E^2 \overset{L}{\geqslant} 0.$$

Furthermore, since $E \overset{L}{\geqslant} 0$, by induction one obtains

$$E^i \overset{L}{\geqslant} E^{i+1} \overset{L}{\geqslant} 0, \qquad i = 1,2,\dots.$$

This shows that the powers of $E$ converge. That $\lim_{i\to\infty} E^i$ is a projection on $N(I-E)$ along $R(I-E)$ follows from [14, Lemma 1] and [1]. ∎

REMARK. Under the conditions of Theorem 4, $BC$ need not be $L$-weakly semipositive, as is shown by taking

$$B=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad C=\begin{pmatrix} 0 & -\frac{1}{4} \\ 0 & 0 \end{pmatrix}$$

and $L=R^2_+$.

In the next section we apply the results of Theorems 1 through 4 and of Corollary 6 to iterative schemes for the system (1.1) obtained from subproper and proper splittings for $A$ [in (1.1)].

## 4. APPLICATIONS TO ITERATIVE METHODS

Let (1.2) be a subproper splitting for $A\in R^{m\times n}$ such that for some solid cone $K\subseteq R^n$,

$$M^+Q=M^+M-M^+A \overset{K}{\geqslant} 0. \tag{4.1}$$

Thus $A$ is a $K$-r.g.s.i. of $M^+$. If the splitting (1.2) is proper, then (as pointed out in Sec. 2) we also have that $M^+$ is a $K$-l.g.s.i. of $A$.

THEOREM 5. *Let $A\in R^{m\times n}$, let $K\subseteq R^n$ be a solid cone, and assume that* (1.2) *is a subproper splitting for $A$ such that $M^+Q\overset{K}{\geqslant}0$. Suppose there exists a vector $k\in K$ with*

$$P_{R(M^T)}k\in K^0 \tag{4.2}$$

*and*

$$M^+Ak\in K. \tag{4.3}$$

*Then*:

(i) $\rho(M^+Q)\leqslant 1$.
(ii) *The Jordan blocks of $M^+Q$ corresponding to $\lambda=1$ are all of order* 1.

*Proof.* Set

$$z\equiv M^+Mk. \tag{4.4}$$

Then $z\in K^0$, and since $N(M)\subseteq N(A)$, we have $M^+Az=M^+Ak$. The conclusions follow now by substituting $B=M^+$, $C=A$, $n=p$, $q=m$ and $L=K$ in Theorem 1. ■

We note that under the conditions of Theorem 5, if the splitting (1.2) is not proper, i.e., $R(A)\subsetneqq R(M)$ and $N(M)\subsetneqq N(A)$, $M^+A$ *cannot* be $K$-semipositive. For, (1.2) and (1.3) yield the factorization

$$A=M(I-M^+Q),$$

and so if $M^+A$ is $K$-semipositive, then $\rho(M^+Q)<1$ (by Theorem 2), which shows that $R(A)=R(M)$. Also, $N(A)=N(M)$ by a dimensionality argument.

In the next three corollaries we consider some applications of Theorem 5 in some special cases. We begin with the "obvious" statement:

COROLLARY 8. *Let* (1.2) *be a subproper splitting for A in* (1.1) *such that if $\lambda$ is an eigenvalue of $M^+Q$ with $|\lambda|=1$, then $\lambda=1$. Then under the conditions of Theorem* 5, *$M^+Q$ is s-convergent for A, and the sequence* (1.5) *converges to a solution of* (1.6) *for all $x_0$.*

A situation described in Corollary 8 arises in the next statement.

COROLLARY 9. *Let $A\in R^{n\times n}$ be symmetric and positive semidefinite, and let* (1.2) *be a subproper splitting for A such that*

$$(Ax,x)\leqslant\omega(Mx,x)\qquad (x \text{ real or complex}),$$

*where $(\cdot,\cdot)$ denotes the usual inner product and where $0\leqslant\omega<2$. If for some solid cone K, $M^+Q\overset{K}{\geqslant}0$ and there exists a vector $k\in K$ such that* (4.2) *and* (4.3) *hold, then $M^+Q$ is s-convergent for A.*

*Proof.* By Theorem 5, $\rho(M^+Q)\leqslant 1$, and the Jordan blocks of $M^+Q$ corresponding to $\lambda=1$ are all of order 1. Let $x\neq 0$ be an eigenvector of $M^+Q$ corresponding to an eigenvalue $\lambda\neq 1,0$. Since

$$M^+Qx=\lambda x,$$

we have $x\in R(M^T)$, and since $R(Q)\subseteq R(M)$ [(1.3)], one obtains from (1.2) that

$$Ax=(1-\lambda)Mx.$$

Hence $0<(Ax,x)=(1-\lambda)(Mx,x) \leqslant \omega(Mx,x)$, and thus $\lambda \geqslant 1-\omega>-1$, whence $|\lambda|<1$ if $\lambda \neq 1$. ∎

REMARK. For the special case where $K=R_+^n$ and where $M$ is nonsingular, Plemmons [14, Theorem 1] shows that if $A$ is positive semidefinite, $M$ is positive definite and $M^{-1}Q \geqslant 0$, then $\rho(M^{-1}Q) \leqslant 1$.

For solvable systems (1.1) we have the next result.

COROLLARY 10. *Suppose that $b \in R(A)$ and that* (1.2) *is a subproper splitting for $A$ in* (1.1) *such that for some cone $K \subseteq R^n$* (4.1), (4.2) *and* (4.3) *hold. If $M^+b \in K$, then there exists a number $\alpha>0$ such that for $x_0 \equiv -\alpha z$, where $z$ is given by* (4.4), *the sequence of iterates $\{x_i\}$ generated by* (1.4) *converges to a solution to* (1.1).

*Proof.* The proof is obtained by constructing sequences of *monotone iterations*.

Because of (4.1), (4.2) and (4.3) we have

$$M^+Qz \overset{K}{\leqslant} z.$$

Let $x$ be some fixed solution to (1.1), and let $\alpha>0$ with $\alpha z+x \in K$. The latter is possible, since $z \in K^0$. Set

$$w_0 \equiv \alpha z+x,$$

and compute $w_1$ from $w_0$ by

$$w_i=M^+Qw_{i-1}+M^+b. \tag{4.5}$$

Now, since $x=M^+Qx+M^+b$ (e.g., [14, Lemma 1]), we have

$$w_1=M^+Q(\alpha z)+x \overset{K}{\leqslant} w_0.$$

Next, since $M^+b \in K$,

$$x_1=-M^+Q(\alpha z)+M^+b \overset{K}{\geqslant} x_0.$$

Moreover,

$$w_0-x_0=2\alpha z+x \in K.$$

Repeated applications of (1.4) and (4.5) and the process of induction yield

$$x_0 \overset{K}{\leqslant} x_1 \overset{K}{\leqslant} \cdots \overset{K}{\leqslant} w_1 \overset{K}{\leqslant} w_0.$$

Thus $\{x_i\}$ is a $K$-monotonically nondecreasing sequence such that $w_0 - x_i \in K$, $i = 0, 1, \dots,$ and so $\{x_i\}$ has a limit (e.g., [3]), say $\bar{x}$. Then

$$\bar{x} = M^+Q\bar{x} + M^+b. \tag{4.6}$$

Now $b \in R(M)$ and $R(Q) \subseteq R(M)$, and so premultiplying both sides of (4.6) by $M$ gives $b = M\bar{x} - N\bar{x} = Ax$. ∎

We come now to iterative schemes induced by a proper splitting of $A$ in (1.1). In [3] it is shown that when such schemes converge, they converge to $A^+b$, the minimum-$l_2$-norm least-squares solution to (1.1).

THEOREM 6. *Let* (1.2) *be a proper splitting for* $A \in R^{m \times n}$ *such that for some solid cone* $K \subseteq R^n$, $M^+Q \overset{K}{\geqslant} 0$. *Then*:

(i) $\rho(M^+Q) < 1$ *and* $P_{R(M^T)}$ *is* $K$-*semipositive if and only if* $M^+A$ *is* $K$-*semipositive.*

(ii) *If there exists a vector* $k \in K$ *such that* (4.2) *and* (4.3) *hold, then* $\rho(M^+Q) < 1$ *and* $M^+Ak_1 \in K^0$ *for some* $k_1 \in K^0$.

*Proof.* Under the conditions of the theorem, $A$ is a $K$-r.g.s.i. of $M^+$.

(i) Since (1.2) is a proper splitting for $A$, we have

$$N(M^{+T}) = N(A). \tag{4.7}$$

The conclusion follows now by substituting $p = n$, $q = m$, $B = M^+$, $C = A$ and $L = K$ in Theorem 2 and noting that (4.7) satisfies (3.10).

(ii) Since (1.2) is a proper splitting for $A$, it follows that

$$R(A) = R(M^{+T}) \tag{4.8}$$

and

$$R(M^+) \cap N(A) = \{0\}. \tag{4.9}$$

Thus for $p = n$, $q = m$, $B = M^+$, $C = A$ and $L = K$ the conditions (4.2), (4.3), (4.8) and (4.9) satisfy, respectively, the conditions (3.2), (3.3), (3.7) and

(3.8), so that $\rho(M^+Q)<1$ by Corollary 6. Moreover, the condition (4.2) shows that $P_{R(M^T)}$ is $K$-semipositive, by Lemma 2(i). Thus $M^+Ak_1 \in K^0$ for some $k_1 \in K^0$, by part (i). ■

REMARK. Under the conditions of the above theorem, if (4.2) and (4.3) hold, then $M^+Ak$ need not be in $K^0$, as we show next. Let

$$K=R^2_+, \qquad A=\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad M=I.$$

Then $M^+Q(K)\subseteq K$, $\rho(M^+Q)<1$, $P_{R(M^T)}\binom{7}{7}\in K^0$, but $M^+A\binom{7}{7}\notin K^0$.

Theorem 6 shows that if (1.2) is a proper and $(K,L)$-weakly regular splitting for $A$ in (1.1) such that (4.2) holds and such that $Ak\in L$, then the sequence (1.4) converges to $A^+b$. In fact if Theorem 6 is combined with Corollary 4, we have the following result, which we state without proof.

COROLLARY 11. *Let* (1.2) *be a proper and* $(K,L)$*-weakly regular splitting for* $A\in R^{m\times n}$ *such that* (4.2) *holds for some* $k\in K$. *Then the following statements are equivalent*:

(i) $A$ *is* $(K,L)$*-monotone.*
(ii) $\rho(M^+Q)<1$.
(iii) $M^+A$ *is* $K$*-semipositive.*

In conclusion, we consider some applications of Theorem 4 to subproper and proper splitting for $A\in R^{m\times n}$.

THEOREM 7. *Let* $K\subset R^n$ *be a solid cone, and let* (1.2) *be a subproper splitting for* $A$ *in* (1.1) *such that* $M^+Q\overset{K}{\geqslant}0$ *and such that* $M^+AM^+Q\overset{K}{\geqslant}0$. *Then the sequence* (1.5) *converges to a solution to* (1.6) *for every* $x_0$. *In the special case when the splitting* (1.2) *is proper, the sequence* (1.5) *(which reduces to the sequence of iterates generated by* (1.4)*) converges to* $A^+b$ *for all* $x_0$.

*Proof.* In the case where (1.2) is a subproper splitting, the proof follows by (4.1) and by substituting $p=n$, $q=m$, $B=M^+$, $C=A$ and $L=K$ in the statement of Theorem 4. When (1.2) is a proper splitting for $A$, 1 is not an eigenvalue of $M^+Q$ (e.g., [3, Lemma 1]); thus $\rho(M^+Q)<1$ and the proof is concluded. ■

COROLLARY 12 (Neumann [13], Plemmons [14]). *Let* (1.2) *be a subproper and regular splitting for* $A \in R^{m\times n}$. *Then* $M^+Q$ *is s-convergent for* $A$ *if*

$$AM^+Q \geqslant 0.$$

*Proof.* Since $M^+ \geqslant 0$, we have $M^+AM^+Q \geqslant 0$. ∎

## 5. CLOSING REMARKS

(1) The problem of obtaining conditions which are necessary and sufficient for $M^+Q$ to be $s$-convergent for $A$ in the presence of a subproper splitting (1.2) for $A$, with $M^+Q \overset{K}{\geqslant} 0$ for some solid cone $K$, remains open. See also [5] and [14].

(2) A study of efficient means of obtaining subproper and proper splittings for a given matrix $A \in R^{m\times n}$ is currently under way. The investigation seems to indicate that, for example, to obtain a proper splitting for $A$, some modifications of an $LU$ decomposition for $A$ lead to computationally effective method for obtaining such a splitting.

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## REFERENCES

1 A. Berman and M. Neumann, Consistency and splittings, *SIAM J. Numer. Anal.* **13** (1976), 877–888.
2 A. Berman and M. Neumann, Proper splittings of rectangular matrices, *SIAM J. Appl. Math.* **31** (1976), 307–312.
3 A. Berman and R. J. Plemmons, Cones and iterative methods for best least squares solutions of linear systems, *SIAM J. Numer. Anal.* **11** (1974), 145–154.
4 A. Berman and R. J. Plemmons, Monotonicity and the generalized inverse, *SIAM J. Appl. Math.* **22** (1972), 155–161.
5 A. Berman and R. J. Plemmons, Eight types of matrix monotonicity, *Linear Algebra Appl.* **13** (1976), 115–123.
6 G. Birkhoff, Linear transformations with invariant cones, *Am. Math. Mon.* **74** (1967), 274–276.
7 E. Bohl, Über eine Zeilensummenbedingung bei $L$-Matrizen, Lecture Notes in Mathematics 395 (Sammelband der Tagung über numerische Lösung nichtlinear

partieller Differential- und Integrodifferentialgleichungen, in Oberwolfach), Springer, 1974, pp. 247–262.
8 D. Carlson, Generalizations of matrix monotonicity, *Linear Algebra Appl.* **13** (1976), 125–131.
9 S. P. Gudder and M. Neumann, Splittings and iterative methods for approximate solutions to singular operator equations in Hilbert spaces, *J. Math. Anal. Appl.*, to be published.
10 V. N. Joshi, A note on the solution of rectangular linear systems by iterations, *SIAM Rev.* **12** (1970), 463–466.
11 H. B. Keller, On the solution of singular and semidefinite systems by iteration, *SIAM J. Numer. Anal.* **2** (1965), 281–290.
12 L. M. Lawson, Computational methods for generalized inverse matrices arising from proper splittings, *Linear Algebra Appl.* **12** (1975), 111–126.
13 M. Neumann, Subproper splittings of rectangular matrices, *Linear Algebra Appl.* **14** (1976), 41–51.
14 R. J. Plemmons, Regular splittings and the discrete Neumann problem, *Numer. Math.* **25** (1976), 153–161.
15 H. S. Price, Monotone and oscillation matrices applied to finite difference approximations, *Math. Comput.* **22** (1968), 489–516.
16 W. C. Rheinboldt and J. S. Vandergraft, A simple approach to the Perron-Frobenius theory for positive operators on general partially-ordered finite-dimensional linear spaces, *Math. Comput.* **27** (1973), 139–145.
17 H. Schneider, Positive operators and an inertia theorem, *Numer. Math.* **7** (1965), 11–17.
18 J. S. Vandergraft, Applications of partial orderings to the study of positive definiteness, monotonicity, and convergence of iterative methods for linear systems, *SIAM J. Numer. Anal.* **9** (1972), 97–104.
19 J. S. Vandergraft, Spectral properties of matrices which have invariant cones, *SIAM J. Appl. Math.* **16** (1968), 1208–1222.
20 R. S. Varga, On recurring theorems on diagonal dominance, *Linear Algebra Appl.* **13** (1976), 1–9.

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